ON THE TORSION OF AN ANISOTROPIC ROD BY FORCES DISTRIBUTED OVER ITS LATERAL SURFACE

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The problem of elastic equilibrium of an isotropic cylinder acted upon by forces applied to its lateral surface and represented by an integer algebraic function of the coordinate measured along the generator has been studied by many authors: Almansi [1], Pearson and Filon [2], Michell [3], Kolosov [4], Zvolinski and Riz [5], Dzhanelidze [6], and others. Some cases of the analogous problem for an anisotropic solid have been discussed by Kosmodamianskii [7], Uzdalev [8] (mainly bending) and Dzhanelidze [9] (general case of anisotropy, loading of general and special character). The question of torsion of anisotropic rods has been the subject of less elaborate studies; only the case of loading uniformly distributed over the length of the rod has been discussed in a paper by Luxenberg [10].

The present contribution deals with the torsion of a rod of rectilinear or cylindrical anisotropy by tangential forces, varying over the length of the rod according to the law of an integer polynomial of the nth degree with respect to z; a general theory is developed and certain special cases are investigated in some detail.

1. General case of torsion of a rod with rectilinear anisotropy. Consider a rod having the shape of a cylinder or prisma, of elastic homogeneous rectilinearly anisotropic material, fixed at one end and carrying tangential loads t distributed over its lateral surface. Assume that:

1) the anisotropy is characterized by the presence of one plane of elastic symmetry normal to the generators;

2) the material obeys Hooke's generalized law and experiences small deformations;

3) body forces are absent.

We locate the origin of the system of coordinates at the centroid of the free end of the rod with the z-axis directed parallel to the generators (Fig. 1). Furthermore, we assume that the loads t are acting at each point tangentially to the boundary line of the cross-section through that point, that, in general, they are distributed along that line non-uniformly, being, however, reducible to a torque and expressible by an integer polynomial of arbitrary degree N with respect to z.

$$t = \sum_{n=0}^{N} t_n(s)\left(\frac{z}{l}\right)^n \qquad (t_n \text{ is a function of the arc } s \text{ of the contour of the cross-section}) \tag{1.1}$$

We shall treat the problem with the same degree of mathematical rigor with which the common torsion problem is being treated; in other words, we shall require that the equations of the problem be fulfilled rigorously on the lateral surface and approximately, by means of integrals, at any cross-section (including the end sections) where we thus confine ourselves to the condition that the acting inner forces be equivalent to a resultant force and moment.

Using the conventional notations for the components of stress, strain and displacement, we may write the system of equilibrium equations of the solid under consideration and the boundary conditions for its lateral surface in the form

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0 \qquad (xyz) \tag{1.2}$$

$$\varepsilon_x = a_{11}\sigma_x + a_{12}\sigma_y + a_{13}\sigma_z + a_{16}\tau_{xy}$$

$$\varepsilon_y = a_{12}\sigma_x + a_{22}\sigma_y + a_{23}\sigma_z + a_{26}\tau_{xy}$$

$$\varepsilon_z = a_{13}\sigma_x + a_{23}\sigma_y + a_{33}\sigma_z + a_{36}\tau_{xy}$$

$$\gamma_{xy} = a_{16}\sigma_x + a_{26}\sigma_y + a_{36}\sigma_z + a_{66}\tau_{xy}$$

$$\gamma_{xz} = a_{55}\tau_{xz} + a_{45}\tau_{yz}, \quad \gamma_{yz} = a_{45}\tau_{xz} + a_{44}\tau_{yz}$$

$$\sigma_x \cos(n, x) + \tau_{xy}\cos(n, y) = -t\cos(n, y)$$

$$\tau_{xy}\cos(n, x) + \sigma_y\cos(n, y) = t\cos(n, x)$$

$$\tau_{xz}\cos(n, x) + \tau_{yz}\cos(n, y) = 0 \tag{1.3}$$

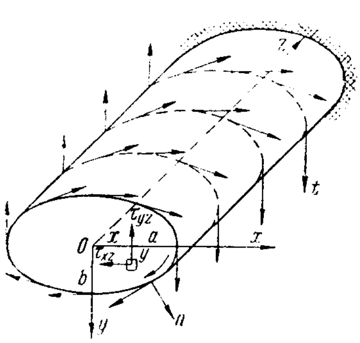

Fig. 1.

where the a_{ij} are elastic constants, while n denotes the normal to the boundary line of a cross-section.

The moment of the external forces acting on the region of the lateral surface between the free end and the cross-section at distance z from the free end is

$$M_z = \sum_{n=0}^{N} \frac{m_n}{(n+1)\,l^n} z^{n+1} \qquad \left(m_n = \int_{\gamma} t_n\,[x\cos(n,\,x) + y\cos(n,\,y)]\,ds\right) \quad (1.4)$$

where m_n is the moment of the forces t_n distributed over the contour γ of the cross-section. For any cross-section the conditions

$$\iint \tau_{xz}\,dxdy = \iint \tau_{yz}\,dxdy = 0, \ \iint (-\tau_{xz}y + \tau_{yz}x)\,dxdy + M_z = 0 \qquad (1.5)$$

$$\iint \sigma_z\,dxdy = \iint \sigma_z x\,dxdy = \iint \sigma_z y\,dxdy = 0 \qquad (1.6)$$

must be fulfilled, with the integrals taken over the cross-sectional area.

In order to investigate the state of stress produced by the loading (1.1) it is obviously sufficient to study the case when the loads are proportional to z^n, where n is an arbitrary integer number, so that we have

$$t = t_n(s)\left(\frac{z}{l}\right)^n \qquad (1.7)$$

For the more general case (1.1) the stresses and displacements will be found by means of superposition.

Starting from Formula (1.7) for the external loading we use for the displacements and stresses hypothetical expressions in the form of sums of decreasing powers of z, namely

$$\begin{aligned} u &= z^{n+2}u_{n+2} + z^n u_n + z^{n-2}u_{n-2} + \ldots \\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ w &= z^{n+1}w_{n+1} + z^{n-1}w_{n-1} + \ldots \end{aligned} \qquad (1.8)$$

$$\begin{aligned} \sigma_x &= z^n\sigma_x^{\,n} + z^{n-2}\sigma_x^{\,n-2} + \ldots \\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \tau_{xy} &= z^n\tau_{xy}^{\,n} + z^{n-2}\tau_{xy}^{\,n-2} + \ldots \\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \tau_{xz} &= z^{n+1}\tau_{xz}^{\,n+1} + z^{n-1}\tau_{xz}^{\,n-1} + \ldots \\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \end{aligned} \qquad (1.9)$$

The quantities appearing in these expressions as multipliers of the powers of z are functions of x, y. The last terms of the sums are

a) in the case of n even

$$u_0, \quad v_0, \quad zw_1; \qquad \sigma_x^{\,0}, \quad \sigma_y^{\,0}, \quad \sigma_z^{\,0}, \quad \tau_{xy}^{\,0}; \qquad z\tau_{xz}^{\,1}, \quad z\tau_{yz}^{\,1}$$

b) in the case of n odd

$$zu_1,\quad zv_1,\quad w_0;\qquad z\sigma_x^1\quad z\sigma_y^1,\quad z\sigma_z^1,\quad z\tau_{xy}^1;\qquad \tau_{xz}^0,\quad \tau_{yz}^0$$

Substituting (1.8) and (1.9) into Equations (1.2) and equating the coefficients of equal powers of z on the left- and the right-hand sides we find

$$u_{n+2}=-\vartheta_{n+2}y+A_{n+2},\qquad v_{n+2}=\vartheta_{n+2}x+B_{n+2}\qquad (1.10.\ n+2)$$

where ϑ, A, B are arbitrary constants and the following systems of equations:

$$\frac{\partial\tau_{xz}^{n+1}}{\partial x}+\frac{\partial\tau_{yz}^{n+1}}{\partial y}=0\qquad (1.10.\ n+1)$$

$$(n+2)(\vartheta_{n+2}x+A_{n+2})+\frac{\partial w_{n+1}}{\partial y}=a_{44}\tau_{yz}^{n+1}+a_{45}\tau_{xz}^{n+1}$$

$$(n+2)(-\vartheta_{n+2}y+B_{n+2})+\frac{\partial w_{n+1}}{\partial x}=a_{45}\tau_{yz}^{n+1}+a_{55}\tau_{xz}^{n+1}$$

$$\frac{\partial\sigma_x^k}{\partial x}+\frac{\partial\tau_{xy}^k}{\partial y}+(k+1)\tau_{xz}^{k+1}=0,\quad \frac{\partial\tau_{xy}^k}{\partial x}+\frac{\partial\sigma_y^k}{\partial y}+(k+1)\ \tau_{yz}^{k+1}=0\ (1.10.\ k)$$

$$\sigma_z^k=\frac{k+1}{a_{33}}w_{k+1}-\frac{1}{a_{33}}(a_{13}\sigma_x^k+a_{23}\sigma_y^k+a_{36}\tau_{xy}^k)$$

$$\frac{\partial u_k}{\partial x}=\beta_{11}\sigma_x^k+\beta_{12}\sigma_y^k+\beta_{16}\tau_{xy}^k+\frac{a_{13}}{a_{33}}(k+1)w_{k+1}$$

$$\frac{\partial v_k}{\partial y}=\beta_{12}\sigma_x^k+\beta_{22}\sigma_y^k+\beta_{26}\tau_{xy}^k+\frac{a_{23}}{a_{33}}(k+1)w_{k+1}$$

$$\frac{\partial v_k}{\partial x}+\frac{\partial u_k}{\partial y}=\beta_{16}\sigma_x^k+\beta_{26}\sigma_y^k+\beta_{66}\tau_{xy}^k+\frac{a_{36}}{a_{33}}(k+1)w_{k+1}$$

$$\left(\beta_{ij}=a_{ij}-\frac{a_{i3}a_{j3}}{a_{33}},\quad i,j=1,2,6;\quad \begin{matrix}k=n,\ n-2,\ \ldots,\ 0\ \text{for}\ n\ \text{even}\\ k=n,\ n-2,\ \ldots,\ 1\ \text{for}\ n\ \text{odd}\end{matrix}\right)$$

$$\frac{\partial\tau_{xz}^{k-1}}{\partial x}+\frac{\partial\tau_{yz}^{k-1}}{\partial y}+k\sigma_z^k=0\qquad (1.10k-1)$$

$$kv_k+\frac{\partial w_{k-1}}{\partial y}=a_{44}\tau_{yz}^{k-1}+a_{45}\tau_{xz}^{k-1},\qquad ku_k+\frac{\partial w_{k-1}}{\partial x}=a_{45}\tau_{yz}^{k-1}+a_{55}\tau_{xz}^{k-1}$$

($k=n,\ n-2,\ \ldots,\ 2$ when n even, and $k=n,\ n-2,\ \ldots,\ 1$ when n odd)

The functions σ_x^m, σ_y^m, ..., τ_{yz}^m satisfy the boundary conditions as well as the conditions which follow from (1.6), and they are to be determined in the region of the cross-section.

2. The general course of the solution of the problem. Equations (1.10) show that the determination of the solution of the problem under discussion reduces to successive solution of two types of problems similar to the problems of simple torsion and of plane strain. For n even we have $1/2(n + 2)$ problems of the first type and the same number of problems of the second type; for n odd it is necessary to solve $1/2(n+3)$ problems of the first type and $1/2(n + 1)$ problems of the second type.

We introduce the following notation:

σ_1^k, σ_2^k, τ^k represent a particular solution of the first two equations (1.10.k);

τ_1^{k-1}, τ_2^{k-1} represent a particular solution of the first equation of the system (1.10.k-1);

$$L_2 = a_{44}\frac{\partial^2}{\partial x^2} - 2a_{45}\frac{\partial^2}{\partial x \partial y} + a_{55}\frac{\partial^2}{\partial y^2}$$

$$L_4 = \beta_{22}\frac{\partial^4}{\partial x^4} - 2\beta_{26}\frac{\partial^4}{\partial x^3 \partial y} + (2\beta_{12} + \beta_{66})\frac{\partial^4}{\partial x^2 \partial y^2} - 2\beta_{16}\frac{\partial^4}{\partial x \partial y^3} + \beta_{11}\frac{\partial^4}{\partial y^4} \qquad (2.1)$$

Furthermore, we introduce the stress functions ψ_{n+1}, ψ_{k-1}, F_k by setting

$$\tau_{xz}^{n+1} = \frac{\partial \psi_{n+1}}{\partial y}, \qquad \tau_{yz}^{n+1} = -\frac{\partial \psi_{n+1}}{\partial x}$$

$$\sigma_x^k = \frac{\partial^2 F_k}{\partial y^2} + \sigma_1^k, \qquad \sigma_y^k = \frac{\partial^2 F_k}{\partial x^2} + \sigma_2^k, \qquad \tau_{xy}^k = -\frac{\partial^2 F_k}{\partial x \partial y} + \tau^k \qquad (2.2)$$

$$\tau_{xz}^{k-1} = \frac{\partial \psi_{k-1}}{\partial y} + \tau_1^{k-1}, \qquad \tau_{yz}^{k-1} = -\frac{\partial \psi_{k-1}}{\partial x} + \tau_2^{k-1}$$

Then we obtain on the basis of (1.10) the following equations for the stress functions:

$$L_2\psi_{n+1} = -2(n+2)\vartheta_{n+2} \qquad (2.3.\mathrm{n}+1)$$

$$\begin{aligned} L_4F_k = {} & \frac{\partial^2}{\partial x^2}\left[\frac{a_{23}}{a_{33}}(k+1)w_{k+1} - \beta_{12}\sigma_1^k - \beta_{22}\sigma_2^k - \beta_{26}\tau^k\right] + \\ & + \frac{\partial^2}{\partial y^2}\left[\frac{a_{13}}{a_{33}}(k+1)w_{k+1} - \beta_{11}\sigma_1^k - \beta_{12}\sigma_2^k - \beta_{16}\tau^k\right] - \\ & - \frac{\partial^2}{\partial x \partial y}\left[\frac{a_{36}}{a_{33}}(k+1)w_{k+1} - \beta_{16}\sigma_1^k - \beta_{26}\sigma_2^k - \beta_{66}\tau^k\right] \end{aligned} \qquad (2.3.\mathrm{k})$$

$$(k = n,\ n-2, \ldots, 0 \ \text{or} \ 1)$$

$$L_2\psi_{k-1} = k\left(\frac{\partial u_k}{\partial y} - \frac{\partial v_k}{\partial x}\right) + \frac{\partial}{\partial x}(a_{45}\tau_1^{k-1} + a_{44}\tau_2^{k-1}) - \frac{\partial}{\partial y}(a_{55}\tau_1^{k-1} + a_{45}\tau_2^{k-1})$$

$$(k = n,\ n-2, \ldots, 2 \ \text{or} \ 1) \qquad (2.3.\mathrm{k}-1)$$

Transformation of the boundary conditions by means of contour integration analogous to that used in the theory of the plane problem and of torsion (see e.g. [11, p. 103]) leads to the following conditions for the boundary line γ:

$$\psi_{n+1} = c_{n+1} \qquad (2.4.n+2)$$

$$\frac{\partial F_n}{\partial x} = -\int_0^s \left(\frac{t_n}{l^n} dy + \sigma_2^n dx - \tau^n dy \right) + c_n'$$

$$\frac{\partial F_n}{\partial y} = \int_0^s \left(\frac{t_n}{l^n} dx + \tau^n dx - \sigma_1^n dy \right) + c_n'' \qquad (2.4.n)$$

$$\frac{\partial F_k}{\partial x} = -\int_0^s (\sigma_2^k dx - \tau^k dy) + c_k', \quad \frac{\partial F_k}{\partial y} = \int_0^s (\tau^k dx - \sigma_1^k dy) + c_k''$$

$$(k = n-2,\ n-4, \ldots, 0 \text{ or } 1) \qquad (2.4.k)$$

$$\psi_{k-1} = \int_0^s (\tau_2^{k-1} dx - \tau_1^{k-1} dy) + c_{k-1} \qquad (k = n,\ n-2, \ldots\ 2 \text{ or } 1) \qquad (2.4.k-1)$$

The integration constants c_{n+1}, c_n', ..., c_{k-1} can be fixed arbitrarily on one of the contours representing the boundary of the multiply connected region of the section; the integrals are taken along the contour between the starting and the current points, thus representing functions of the arc s.

The conditions (1.6) at the cross-section resolve into

$$\iint \tau_{xz}^{n+1} dxdy = \iint \tau_{yz}^{n+1} dxdy = 0, \quad \iint (-\tau_{xz}^{n+1} y + \tau_{yz}^{n+1} x)\, dxdy + \frac{m_n}{(n+1)\, l^n} = 0 \qquad (2.5.\ n+1)$$

$$\iint \tau_{xz}^{k-1} dxdy = \iint \tau_{yz}^{k-1} dxdy = 0, \quad \iint (-\tau_{xz}^{k-1} y + \tau_{yz}^{k-1} x)\, dxdy = 0 \qquad (2.5.\ k-1)$$

$$(k = n,\ n-2, \ldots, 2 \text{ or } 1)$$

$$\iint \sigma_z^k dxdy = \iint \sigma_z^k x dxdy = \iint \sigma_z^k y dxdy = 0 \quad (k = n,\ n-2, \ldots, 0 \text{ или } 1) \qquad (2.5.\ k)$$

Consider, in particular, the case when the cross-sectional domain is simply connected. For the latter we may assume that $\psi_{n+1} = 0$ along the contour. Then the following course of solution of the problem may be contemplated for an arbitrary integer $n > 1$:

1. Solve the problem of simple torsion, i.e. determine ψ_{n+1}, τ_{xz}^{n+1}, τ_{yz}^{n+1} leading to

$$\psi_{n+1} = (n+2)\, \vartheta_{n+2} \psi(x, y) \qquad (2.6)$$

The constant ϑ_{n+2} follows from the third equation (2.5. n+1), leading to

$$\vartheta_{n+2} = -\frac{m_n}{(n+2)(n+1)Cl^n} \qquad \left(C = 2\iint \psi\, dxdy\right) \tag{2.7}$$

where C denotes the rigidity in the usual sense (i.e. for a like anisotropic rod twisted by moments applied to its plane ends). The first two conditions (2.5. n+1) will be fulfilled identically, because $\psi = 0$ along the contour.

2. Determine w_{n+1} and σ_1^n, σ_2^n, τ^n from Equations (1.10. n+1) and (1.10. n), leading to

$$w_{n+1} = W_{n+1} - (n+2)(A_{n+2}x + B_{n+2}y + C_{n+1}) \tag{2.8}$$

where W_{n+1} is a function free of indeterminate constants, while C_{n+1} is a constant of integration.

3. Solve the plane problem, i.e. find σ_x^n, σ_y^n, τ_{xy}^n and σ_z^n. The first three functions will be free of indeterminate elements, while

$$\sigma_z^n = \sigma_{z0}^n - \frac{(n+2)(n+1)}{a_{33}}(A_{n+2}x + B_{n+2}y + C_{n+1}) \tag{2.9}$$

The three arbitrary constants will be found from the conditions (2.5. n) for σ_z^n.

4. Determine u_n, v_n from (1.10. n) as well as τ_1^{n-1}, τ_2^{n-1}; we find

$$u_n = U_n - \vartheta_n y + A_n, \qquad v_n = V_n + \vartheta_n x + B_n \tag{2.10}$$

where U_n, V_n are known functions, while ϑ_n, A_n, B_n are new arbitrary constants.

5. Determine ψ_{n-1}, τ_{xz}^{n-1}, τ_{yz}^{n-1} from (2.3. n−1), leading to

$$\psi_{n-1} = 2n\vartheta_n\psi + \Psi_{n-1} \tag{2.11}$$

$$\tau_{xz}^{n-1} = 2n\vartheta_n\frac{\partial\psi}{\partial y} + r_{n-1}, \qquad \tau_{yz}^{n-1} = -2n\vartheta_n\frac{\partial\psi}{\partial x} + s_{n-1} \tag{2.12}$$

where ψ, Ψ_{n-1}, r_{n-1}, s_{n-1} are known functions; of the conditions (2.5. n−1) the first two are fulfilled identically, while the third gives

$$\vartheta_n = \frac{1}{nC}\iint (r_{n-1}y - s_{n-1}x)dx\,dy \tag{2.13}$$

Then we find

1) w_{n-1}, $\sigma_1^{\;n-2}$, $\sigma_2^{\;n-2}$, τ^{n-2}; 2) $\sigma_x^{\;n-2}$, $\sigma_y^{\;n-2}$, $\tau_{xy}^{\;n-2}$, $\sigma_z^{\;n-2}$;

3) u_{n-2}, v_{n-2}, $\tau_1^{\;n-3}$, $\tau_2^{\;n-3}$; 4) ψ_{n-3}, $\tau_{xz}^{\;n-3}$, $\tau_{yz}^{\;n-3}$ etc..

The arbitrary constants are to be determined from the conditions (2.5. k-1) and (2.5.k).

Having arrived at the last terms of the sums (1.8) and (1.9) we obtain three indeterminate constants representing "rigid" displacements: ϑ_0, A_0, B_0 (in the case of n even) or A_1, B_1, C_0 (for the case of n odd). Adding to the displacements the missing terms representing "rigid" displacements and containing three constants, we find all six constants from the constraint conditions of an element of the terminal cross-section.

The same order of operations in the process of solution is preserved in the case of a cross-section of multiple connection, with the difference that the formulas for ϑ_k become more complicated.

Adding to the obtained stresses and displacements the solution for the case of a rod with a free lateral surface acted upon by moments M' applied to the end sections, we can obtain the solution for a rod with both ends fixed; the unknown moment reaction M' is then to be determined from the constraint conditions of the end $z = 0$.

Entirely analogous is the procedure of solution for the torsion problem of a rod of cylindrical anisotropy, with an axis of anisotropy parallel to the generators and a plane of elastic symmetry normal to that axis. In this case we have to start from the fundamental system of equations in cylindrical coordinates analogous to (1.2)

$$\frac{\partial\sigma_r}{\partial r}+\frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta}+\frac{\partial\tau_{rz}}{\partial z}+\frac{\sigma_r-\sigma_\theta}{r}=0$$

$$\varepsilon_r = a_{11}\sigma_r + a_{12}\sigma_\theta + a_{13}\sigma_z + a_{16}\tau_{r\theta}\ \ldots . \tag{2.14}$$

$$\gamma_{r\theta} = a_{16}\sigma_r + a_{26}\sigma_\theta + a_{36}\sigma_z + a_{66}\tau_{r\theta},\ \ldots .\qquad \gamma_{rz} = a_{55}\tau_{rz} + a_{45}\tau_{\theta z}$$

The axis of anisotropy is used here as the z-axis. The order of the steps for determination of the unknown functions and constants remains the same as in the case of rectilinear anisotropy; it leads to equations of the second and fourth order for the stress functions ψ_{k-1}, F_k, similar to the equations of the theory of torsion and plane strain (see [11, pp.179, 182, 201]), which must be solved consecutively.

3. Torsion of a rod of elliptical cross-section. The treatment of actual problems may be simplified by use of complex representation of stress functions, stresses and displacements by means of functions $\Phi_{jk}(z_j)$ of the complex variables $z_j = x + \mu_j y$ ($j = 1, 2, 3$; $k = n, n-2, \ldots, 0$ or 1), as in the theory of torsion, and of the plane problem ([11, pp. 112,150]). The problem is then reduced to the determination of the functions $\Phi_{jk}(z_j)$ in the region of the cross-section (the total number of the functions is $1/2(3n+6)$ if n is even and $1/2(3n+5)$ if n is odd); these functions must fulfil known boundary conditions along the contour, and the number of these conditions secures correctness in the statement of the problem and the uniqueness of its solution.

If the rod has the shape of an elliptical cylinder (Fig. 1) and the loading is uniformly distributed along the contour of each cross-section so that

$$t = t_n\left(\frac{z}{l}\right)^n \tag{3.1}$$

where t_n = const, then the solution of the problem is elementary for any degree n in terms of integer polynomials. The functions ψ_{k-1}, F_k become integer polynomials of degree $n + 4 - k$; the coefficients of the polynomials are determined by Equations (2.3), the boundary conditions and the conditions (2.5. k − 1) of the cross-sections. We give here the expressions for the constants and the coefficients of the first terms in the expressions for the stresses and the displacement w of an orthotropic rod ($a_{16} = a_{26} = a_{36} = a_{45} = 0$), one end of which is built-in:

$$\vartheta_{n+2} = -\frac{2t_n}{a^2(n+2)(n+1)\,l^n}(a_{44} + a_{55}c^2), \qquad A_{n+2} = B_{n+2} = C_{n+1} = 0 \tag{3.2}$$

$$\tau_{xz}^{\ n+1} = \frac{4t_n}{(n+1)\,l^n}\frac{y}{b^2}, \qquad \tau_{yz}^{\ n+1} = -\frac{4t_n}{(n+1)\,l^n}\frac{x}{a^2} \tag{3.3}$$

$$\sigma_x^{\ n} = -\frac{2t_n}{b^2 l^n}xy, \qquad \sigma_y^{\ n} = \frac{2t_n}{a^2 l^n}xy, \qquad \tau_{xy}^{\ n} = \frac{t_n}{l^n}\left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right) \tag{3.4}$$

$$\sigma_z^{\ n} = \frac{2t_n}{a^2 a_{33} l^n}\left[(a_{13} + a_{55})c^2 - a_{23} - a_{44}\right]xy, \qquad w_{n+1} = \frac{2t_n}{a^2(n+1)\,l^n}(a_{55}c^2 - a_{44})\,xy \tag{3.5}$$

In these formulas a and b denote the semi-axes of the ellipse and $c = a/b$.

In the particular case of a uniformly loaded rod we obtain the known result [10]

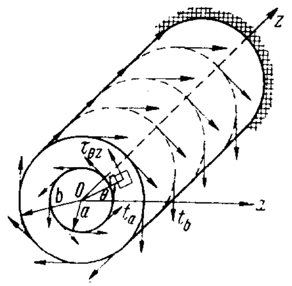

Fig. 2.

$$\tau_{xz} = \frac{4t_0}{b^2} yz, \quad \tau_{yz} = -\frac{4t_0}{a^2} xz \tag{3.6}$$

$$\sigma_x = -\frac{2t_0}{b^2} xy, \quad \sigma_y = \frac{2t_0}{a^2} xy, \quad \tau_{xy} = t_0\left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right)$$

$$\sigma_z = \frac{2t_0}{a^2 a_{33}} \left[(a_{13} + a_{55})\, c^2 - a_{23} - a_{44}\right] xy \tag{3.7}$$

The stress components, with the exception of σ_z, are in this case independent of the elastic constants; in other words, they are, respectively, identical to the corresponding components for the isotropic rod.

4. Torsion of a hollow cylinder by symmetrically distributed loads. Consider a hollow circular cylinder (tube) characterized by cylindrical anisotropy with an axis z of anisotropy coinciding with the axis of the cylinder. Assume that one of its ends is fixed and the other is free, while twisting tangential forces are applied to the cylindrical surfaces (Fig. 2), these loads being proportional to a certain power of z and uniformly distributed along the contour of every cross section, so that

$$t_a = t_{na}\left(\frac{z}{l}\right)^n, \qquad t_b = t_{nb}\left(\frac{z}{l}\right)^n \qquad (t_{na},\ t_{nb} \text{ are constant coefficients}) \tag{4.1}$$

This problem is easily solved if there is one plane of elastic symmetry at each point - normal or radial; however, in order to avoid simple but cumbersome computations we shall concentrate on the case of an orthotropic cylinder ($a_{16} = a_{26} = a_{36} = a_{45} = 0$ in Equations (2.14)).

We introduce the following notations: a, b are inner and outer radii of the cross-section, respectively, $c = a/b$, l is the length of the rod, $G_{\theta z} = 1/a_{44}$ is the shear modulus corresponding to changes of angles between direction θ and the axial direction, i.e. for planes parallel to the axis and normal to a radius of a cross-section, $G_{r\theta} = 1/a_{66}$ is the shear modulus corresponding to changes of angles between the directions r and θ in the planes of the cross-sections

$$g = G_{\theta z} / G_{r\theta}$$

In the case under consideration we may put

$$\sigma_r = \sigma_\theta = \sigma_z = \tau_{rz} = 0, \qquad u = w = 0 \tag{4.2}$$

The fundamental system (2.14) assumes the form

$$\frac{\partial}{\partial r}(r^2\tau_{r\theta}) + \frac{\partial}{\partial z}(r^2\tau_{\theta z}) = 0, \quad G_{\theta z}\frac{\partial v}{\partial z} = \tau_{\theta z}, \quad G_{r\theta}\left(\frac{\partial v}{\partial z} - \frac{v}{r}\right) = \tau_{r\theta} \tag{4.3}$$

In these systems of equations u, v, w are the projections of the displacement on the directions r, θ, z, respectively.

We seek the displacement v and the stresses in the form of sums

$$v = z^{n+2}v_{n+2} + z^n v_n + z^{n-2}v_{n-2} + \ldots \tag{4.4}$$

$$\tau_{r\theta} = z^n \tau_{r\theta}{}^n + z^{n-2}\tau_{r\theta}{}^{n-2} + \ldots, \qquad \tau_{\theta z} = z^{n+1}\tau_{\theta z}{}^{n+1} + z^{n-1}\tau_{\theta z}{}^{n-1} + \ldots \tag{4.5}$$

The last terms in the sums will be v_0, $\tau_{r\theta}$, $z\tau^1_{\theta z}$ in the case of n even and zv_1, $z\tau^1_{r\theta}$, $\tau^0_{\theta z}$ in the case of n odd. The coefficients of z^m are functions of r only.

Substituting these expressions into Equations (4.3) we obtain

$$v_{n+2} = \vartheta_{n+2}r \tag{4.6. n+2}$$

$$\tau_{\theta z}{}^{n+1} = (n+2)\vartheta_{n+2}G_{\theta z}r \tag{4.6. n+1}$$

$$\tau_{r\theta}{}^k = \frac{G_{\theta z}B_k}{r^2} - \frac{k+1}{r^2}\int r^2\tau_{\theta z}{}^{k+1}dr, \quad v_k = \vartheta_{,k}r + \frac{r}{G_{r\theta}}\int\frac{\tau_{r\theta}{}^k}{r}dr \tag{4.6. k}$$

$(k = n,\ n-2, \ldots, 0$ or $1)$

$$\tau_{\theta z}{}^{k-1} = kG_{\theta z}v_k \qquad (k = n,\ n-2, \ldots, 2 \text{ or } 1) \tag{4.6. k-2}$$

The conditions for the outer and inner surfaces of the cylinder are

$$\begin{aligned} \tau_{r\theta}{}^n &= t_{nb}/l^n, \quad \tau_{r\theta}{}^k = 0 \quad \text{when } r = b \\ \tau_{r\theta}{}^n &= t_{na}/l^n, \quad \tau_{r\theta}{}^k = 0 \quad \text{when } r = a \end{aligned} \tag{4.7}$$

The conditions for the plane end faces (and for any cross-section) lead to the relations

$$\int_a^b \tau_{\theta z}{}^{n+1}r^2dr = \frac{t_{na}a^2 - t_{nb}b^2}{l^n(n+1)}, \qquad \int_a^b \tau_{\theta z}{}^{k-1}r^2dr = 0 \tag{4.8}$$

Knowing the type of the function $\tau^{n+1}_{\theta z}$ we find from Equations (4.6.k) consecutively the functions $\tau_{r\theta}{}^n$, v_n, $\tau_{\theta z}{}^{n-1}$, $\tau_{r\theta}{}^{n-2}$, v_{n-2}, $\tau_{\theta z}{}^{n-3}$, and so forth. The determination of the unknown constants ϑ_k and B_k takes place with the aid of the conditions (2.7) which permit all constants except one to be found, namely ϑ_0 for n even or ϑ_1 for n odd; the former, expressing a "rigid" displacement (rotation), will be found from the condition for the fixed end, the latter from the second condition (4.8) for $\tau_{\theta z}{}^0(k = 1)$. In the case of n odd it will be necessary for the sake of definiteness, to add $v' = \vartheta_0{}'r$ to the displacement; the constant $\vartheta_0{}'$ will be determined from the condition for the fixed end. The simple structure of Equations (4.6.k) permits the construction of general expressions for displacement and stresses corresponding to an arbitrary n. In the case of a tube these expressions become quite cumbersome; we give here only the first two or three terms of the sums:

$$v = \vartheta_{n+2} r z^{n+2} + \left[\vartheta_n r - \frac{(n+2)(n+1)}{4.2}\vartheta_{n+2} g r^3 - \frac{B_n g}{2r}\right] z^n + \tag{4.9}$$
$$+ \left[\vartheta_{n-2} r - \frac{n(n-1)}{4.2}\vartheta_n g r^3 + \frac{(n+2)(n+1)n(n-1)}{6.4.4.2}\vartheta_{n+2} g^2 r^5 + \right.$$
$$\left. + \frac{n(n-1)B_n g^2}{4} r \ln r - \frac{B_{n-2} g}{2r} z^{n-2}\right] + \ldots$$

$$\tau_{\theta z} = G_{\theta z}\left\{(n+2)\vartheta_{n+2} r z^{n+1} + \right.$$
$$\left. + n\left[\vartheta_n r - \frac{(n+2)(n+1)}{4.2}\vartheta_{n+2} g r^3 - \frac{B_n g}{2r}\right] z^{n-1} + \ldots\right\}$$

$$\tau_{r\theta} = G_{\theta z}\left\{\left[-\frac{(n+2)(n+1)}{4}\vartheta_{n+2} r^2 - \frac{B_n}{r^2}\right] z^n + \left[-\frac{n(n-1)}{4}\vartheta_n r^2 + \right.\right.$$
$$\left.\left. + \frac{(n+2)(n+1)n(n-1)}{6.4.2}\vartheta_{n+2} g r^4 + \frac{n(n-1)B_n g}{4} + \frac{B_{n-2}}{r^2}\right] z^{n-2} + \ldots\right\}$$

In the case of a solid rod of circular cross-section ($a = 0$) displacement and stresses can be represented for an arbitrary integer n by the expressions

$$v = \vartheta_{n+2} r z^{n+2} + \sum_{k=0}\left[\vartheta_{n-2k} r + \sum_{m=1}^{k+1}(-1)^m\left(\frac{g}{4}\right)^m p^n_{mk}\vartheta_{n+2m-2k} r^{2m+1}\right] z^{n-2k}$$

$$\tau_{\theta z} = G_{\theta z}\left\{(n+2)\vartheta_{n+2} r z^{n+1} + \sum_{k=0}^{k+1}(n-2k)\left[\vartheta_{n-2k} r + \right.\right. \tag{4.10}$$
$$\left.\left. + \sum_{m=1}^{k+1}(-1)^m\left(\frac{g}{4}\right)^m p^n_{mk}\vartheta_{n+2m-2k} r^{2m+1}\right] z^{n-2k-1}\right\}$$

$$\tau_{r\theta} = 0.5\, G_{\theta z}\left\{-p^n_{10}\vartheta_{n+2} r^2 z^n + \right.$$
$$\left. + \sum_{k=1}\left[\sum_{m=1}^{k+1}(-1)^m\left(\frac{g}{4}\right)^{m-1} m\, p^n_{mk}\vartheta_{n+2m-2k} r^{2m}\right] z^{n-2k}\right\}$$

where

$$p^n_{mk} = \frac{(n+2m-2k)(n+2m-2k-1)\cdots(n+1-2k)}{(m+1)!\,m!} = \frac{1}{m}\binom{n+2m-2k}{2m}\binom{2m}{m-1}$$
$$(m = 1, 2, 3, \ldots;\ n, k = 0, 1, 2, 3\ldots) \tag{4.11}$$

The upper limits of summation in the first two sums of Formulas (4.10) are $1/2n$ in the case of even n and $1/2(n-1)$ in the case of odd n.

If the free end of the hollow or solid cylinder is acted upon by a twisting moment M', then to the displacements and stresses (4.9) or (4.10) one must add

$$v' = \frac{2M'}{G_{\theta z}\pi(b^4-a^4)} rz + \vartheta_0' r, \qquad \tau_{\theta z}' = \frac{2M'}{\pi(b^4-a^4)} r, \qquad \tau_{r\theta}' = 0 \tag{4.12}$$

respectively. If the rod is fixed at both ends, then the unknown moment reaction M' and the rotation $\vartheta_0 + \vartheta_0'$ (or ϑ_0') are to be determined from the conditions at the end faces. If we assume that the outer contours of the plane end faces are fixed, then the conditions mentioned are

$$v(b, 0) = v(b, l) = 0 \tag{4.13}$$

5. Particular cases of distribution of torsional loading. Let us consider in greater detail the case, discussed in Section 4, when the inner surface of the hollow cylinder is not subjected to loads, while the loading, applied to the outer surface, is distributed according to the quadratic law

$$t = t_0\left(\alpha_0 + \alpha_1\frac{z}{l} + \alpha_2\frac{z^2}{l^2}\right) \tag{5.1}$$

where a_0, a_1, a_2 are given dimensionless coefficients. This loading reduces to a twisting moment

$$M = \frac{1}{3}\pi t_0 b^2 l(6\alpha_0 + 3\alpha_1 + 2\alpha_2), \qquad \text{or} \qquad t_0 = \frac{3M}{\pi b^2 l}\,\frac{1}{6\alpha_0 + 3\alpha_1 + 2\alpha_2} \tag{5.2}$$

We give in the following the formulas for displacements and stresses obtained on the basis of (4.9).

Case 1. One end fixed, the other free.

$$\begin{aligned} v = \frac{t_0 b^2}{G_{\theta z}(b^4-a^4)}\Bigg\{&\left[2(l^2-z^2)\alpha_0 + \frac{2\alpha_1}{3l}(l^3-z^3) + \frac{\alpha_2}{3l^2}(l^4-z^4)\right]r + \\ &+\frac{g}{2}\left[\left(\alpha_0+\alpha_1\frac{z}{l}+\alpha^2\frac{z^2}{l^2}\right)\left(r^3+\frac{a^4}{r}\right) - (\alpha_0+\alpha_1+\alpha_2)\left(b^3+\frac{a^4}{b}\right)\frac{r}{b} + \right.\\ &\left.+\frac{2}{3}\,\frac{b^4+a^2b^2+4a^4}{b^2+a^2}\left(\alpha_1+\alpha_2-\alpha_1\frac{z}{l}-\alpha_2\frac{z^2}{l^2}\right)r\right] + \\ &+\frac{g^2}{24(b^2+a^2)}\,\frac{\alpha_2}{l^2}\left[2(b^4+a^2b+4a^4)(r^3-b^2r) + 2a^4(b^2-3a^2)\left(\frac{b^2}{r}-r\right) + \right.\\ &\left.+(b^2+a^2)(b^4r-r^5) + 12a^4(b^2+a^2)r\ln\frac{b}{r}\right]\Bigg\} \end{aligned} \tag{5.3}$$

$$\tau_{\theta z} = \frac{t_0 b^2}{b^4 - a^4}\left\{-2\left(2\alpha_0 z + \alpha_1 \frac{z^2}{l} + \frac{2\alpha_2}{3}\frac{z^3}{l^2}\right) r + \right.$$
$$\left. + \frac{g}{2l}\left(\alpha_1 + \frac{2\alpha_2 z}{l}\right)\left(r^3 + \frac{a^4}{r}\right) - g\, \frac{b^4 + a^2 b^2 + 4a^4}{3(b^2 + a^2)}\left(\frac{\alpha_1}{l} + \frac{2\alpha_2 z}{l}\right) r\right\} \quad (5.4)$$
$$\tau_{r\theta} = \frac{t_0 b^2}{b^4 - a^4}\left\{\left(\alpha_0 + \alpha_1 \frac{z}{l} + \alpha_2 \frac{z^2}{l^2}\right)\left(r^2 - \frac{a^4}{r^2}\right) + \right.$$
$$\left. + g\, \frac{\alpha_2}{6l^2(b^2 + a^2)}\left[(b^4 + a^2 b^2 + 4a^4) r^2 + \frac{a^4 b^2 (3a^2 - b^2)}{r^2} - (b^2 + a^2)(r^4 + 3a^4)\right]\right\}$$

The twist angle in any cross-section is a variable quantity depending on r and z; we shall call the twist angle for a given cross-section the angle by which the outer contour of the section rotates:

$$\varphi = \frac{v(b,\ z)}{b} \quad (5.5)$$

In the case under consideration we obtain the maximum twist angle at the free end:

$$\varphi_{\max} = \frac{Ml}{\pi G_{\theta z} b^4 (1 - c^4)}\, \frac{6\alpha_0 + 2\alpha_1 + \alpha_2}{6\alpha_0 + 3\alpha_1 + 2\alpha_2}\left[1 - \frac{g}{2}\, \frac{\alpha_1 + \alpha_2}{6\alpha_0 + 2\alpha_1 + \alpha_2}\left(\frac{b}{l}\right)^2 f(c)\right] \quad (5.6)$$

where

$$f(c) = \frac{(1 + 3c^2)(1 - c^2)^2}{1 + c^2}, \qquad c = \frac{a}{b} \quad (5.7)$$

The maximum stress appears at the outer surface of the fixed end:

$$\tau_{\max} = |\tau_{\theta z}(b,\ l)| = \frac{2M}{\pi b^3 (1 - c^4)}\left[1 - \frac{g}{4}\, \frac{\alpha_1 + 2\alpha_2}{6\alpha_0 + 3\alpha_1 + 2\alpha_2}\left(\frac{b}{l}\right)^2 f(c)\right] \quad (5.8)$$

The complete twist angle and the maximum stress in a rod with its cylindrical surface free, but twisted by moments of the same amount M applied at its ends are, as is well-known, respectively equal to

$$\varphi_M = \frac{2Ml}{\pi G_{\theta z} b^4 (1 - c^4)} \qquad \tau_M = \frac{2M}{\pi b^3 (1 - c^4)} \quad (5.9)$$

These formulas permit comparison of distributed and concentrated twisting loads with each other and an estimate of the changes in $\phi_{\max}$ and $\tau_{\max}$ which take place if the moments, acting at the ends, are replaced by distributed loading.

Case 2. Both ends fixed.

$$v = \frac{t_0 b^2}{G_{\theta z}(b^4 - a^4)}\left\{\left[2\alpha_0 (lz - z^2) + \frac{2\alpha_1}{3l}(l^2 z - z^3) + \frac{\alpha_2}{3l^2}(l^3 z - z^4)\right] r + \right.$$
$$+ \frac{g}{2}\left[\left(\alpha_0 + \alpha_1 \frac{z}{l}\right)\left(r^3 + \frac{a^4}{r} - b^2 r - \frac{a^4 r}{b}\right) + \alpha_2 \frac{z^2}{l^2}\left(r^3 + \frac{a^4}{r}\right) - \right.$$

$$-\frac{\alpha^2}{3l^2}\,\frac{2b^2(b^4+a^2b^2+4a^4)z^2+(b^2+3a^2)(b^2-a^2)^2\,lz}{b^2(b^2+a^2)}\Bigg]r+$$

$$+\frac{g^2\alpha_3}{24(b^2+a^2)l^2}\Big[2(b^4+a^2b^2+4a^4)(r^3-b^2r)+$$

$$+2a^4(b^2-3a^2)\left(\frac{b^2}{r}-r\right)-(b^2+a^2)(b^4r-r^5)+12a^4(b^2+a^2)r^2\ln\frac{b}{r}\Big]\Big\} \qquad (5.10)$$

The stress $\tau_{r\theta}$ is again determined by Formula (5.4) given above, while $\tau_{\theta z}$ is to be obtained by differentiation of v (4.3).

If a rod with fixed ends is acted upon by a concentrated torque M applied to it at its center cross-section, then the maximum twist angle (at the central cross-section) and the maximum stress are

$$\varphi_M{}'=\frac{Ml}{2\pi G_{\theta z}b^4(1-c^4)}, \qquad \tau_M{}'=\frac{M}{\pi b^3(1-c^4)} \qquad (5.11)$$

(as in the case of a rod of length 1/2 l twisted by a torque of magnitude 1/2 M).

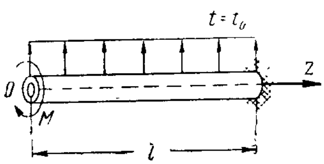

Fig. 3.

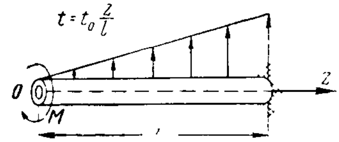

Fig. 4.

Here are a few particular cases.

1. Rod with one end fixed, acted upon by loads uniformly distributed over its length (Fig. 3)

$$\alpha_0=1, \qquad \alpha_2=\alpha_3=0, \qquad \varphi_{\max}=0.5\varphi_M, \qquad \tau_{\max}=\tau_M \qquad (5.12)$$

The same results are obtained for rods with both ends fixed.

2. Rod with one end fixed, loaded linearly along its length (Fig. 4).

If the fixed end is $z = l$, then $\alpha_0 = \alpha_2 = 0$, $\alpha_1 = 1$

$$\varphi_{\max}=\frac{1}{3}\left[1-\frac{g}{4}\left(\frac{b}{l}\right)^2 f(c)\right]\varphi_M \quad \tau_{\max}=\left[1-\frac{g}{12}\left(\frac{b}{l}\right)^2 f(c)\right]\tau_M \qquad (5.13)$$

If the fixed end is $z = 0$, while the other end is free (Fig. 4), then, of course, both the twist angle and the stress are larger than in the case of the end $z = l$ being fixed, and we find

$$\varphi_{\max} = \frac{2}{3}\left[1 + \frac{g}{8}\left(\frac{b}{l}\right)^2 f(c)\right]\varphi_M, \qquad \tau_{\max} = \left[1 + \frac{g}{12}\left(\frac{b}{l}\right)^2 f(c)\right]\tau_M \quad (5.14)$$

In the case of a long thin rod the ratio b/l is a small quantity, and if the ratio g of the shear moduli is small or comparable with unity, the second terms in Formulas (5.13) and (5.14) can be neglected, and we then find

$$\tau_{\max} = \tau_M \text{ and } \varphi_{\max} = {}^1/_3\, \varphi_M \text{ or } \varphi_{\max} = {}^2/_3\, \varphi_M$$

3. Rod with both ends fixed, carrying loads distributed according to the parabolic law (Fig. 5).

$$t = \frac{4t_0}{l^2}(lz - z^2) \qquad (5.15)$$

Here we have $a_0 = 0$, $a_1 = -a_2 = 4$ and

$$\varphi_{\max} = \frac{5}{8}\left[1 + \frac{2}{5}g\left(\frac{b}{l}\right)^2 f(c)\right]\varphi'_M$$

$$\tau_{\max} = \left[1 + \frac{1}{2}g\left(\frac{b}{l}\right)^2 f(c)\right]\tau_M' \qquad (5.16)$$

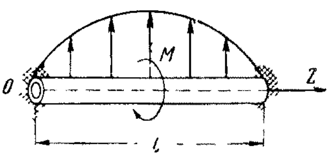

Fig. 5.

In the case of a solid cylinder we have to use the same formulas after having substituted into them $c = 0$, $f = 1$.

Take as a further example the case of a solid cylinder with one end fixed and carrying the load

$$t = t_6\left(\frac{z}{l}\right)^6 \qquad (5.17)$$

Using the general expressions (4.10), (4.11) with $n = 6$ we determine from the boundary conditions consecutively the constants $\vartheta_8, \vartheta_6, \vartheta_4, \vartheta_2$ and from the constraint condition the constant ϑ_0. The ultimate result is of the form

$$v = -\frac{t_6}{G_{\theta z} b^2 l^6}\left\{\frac{r}{14}(z^8 - l^8) \; [\text{illegible}] \; \left(\frac{g}{4}\right)\left[(2b^2r - 3r^3)z^6 + b^3l^6\right] + \right.$$
$$+ \frac{5}{3}\left(\frac{g}{4}\right)^2 [(5b^4r - 12b^2r^{[\text{illegible}]} \; [\text{illegible}] \, r^5)z^4 + b^5l^4] + \frac{2}{3}\left(\frac{g}{4}\right)^3 [(26b^6r - 75b^4r^3 +$$
$$\left. + 60b^2r^5 - 15r^7)z^2 + 4b^7l^2] + \frac{1}{3}\left(\frac{g}{4}\right)^4(-52b^6r^3 + 50b^4r^5 - 20b^2r^7 + 3r^9 + 19b^9)\right\}$$

$$\tau_{r\theta} = \frac{t_6}{l^6}\left(\frac{r}{b}\right)^2\left\{z^6 + \frac{g}{4}(b^2 - r^2)\left[10z^4 + 5\left(\frac{g}{4}\right)(5b^2 - 3r^2)z^2 + \right.\right.$$
$$\left.\left. + \frac{2}{3}\left(\frac{g}{4}\right)^2(13b^4 - 12b^2r^2 + 3r^4)\right]\right\} \qquad (5.18)$$

$$\tau_{\theta z} = G_{\theta z}\frac{\partial v}{\partial z} \qquad (5.19)$$

All given formulas become formulas for an isotropic rod if we substitute $G_{\theta z} = G$, $g = 1$.

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